AN OPTIMAL MAINTENANCE POLICY FOR A SYSTEM UNDER PERIODIC OVERHAUL

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Abstract

A preventive maintenance policy with minimal repair at failure, periodic overhaul and replacement is considered. A model describing the effect of overhaul is proposed and the expected cost rate is obtained under negligible or non-negligible maintenance time. Based on this model, the optimal number of overhauls and optimal interval between overhauls for minimizing the expected cost rate over infinite time horizon are determined. Under some mild conditions, a unique optimal maintenance policy exists and a closed form expression for the optimal interval between overhauls is derived when the lifetime of the system follows Weibull distribution. Numerical studies are made.

1. Introduction

Maintenance extends equipment lifetime or at least the mean time to failure, and an effective maintenance policy can reduce the frequency of failures and the undesirable consequences of such failures. Maintenance clearly impacts on component and system reliability; too little maintenance may result in an excessive number of costly failures and poor system performance, and therefore, reliability decreases; excessive maintenance may improve reliability, but the maintenance cost will be sharply increased. Therefore, a maintenance policy which holds the balance of the two expenditures is necessary.

Maintenance can be categorized into two classes: corrective and preventive ones. Corrective maintenance (CM), called repair, is all actions performed to restore the system to functioning condition when it fails. Preventive maintenance (PM) is all actions performed to prevent failures when the system is operating. In practice, a system that deteriorates with age usually receives repair and two PM operations: replacement and overhaul. A replacement is to restore the system to its original state as a new one and an overhaul is performed to eliminate any impending failure of the system.

Most previous studies on PM policies assume that the system is restored to as good as new state after PM. Pierskalla and Voelker [8] and Sherif and Smith [9] provide detailed surveys on PM policies. Since an overhaul may affect only a limited number of components, the overhaul makes a system 'better than old' but not as good as new. Recently, Liu et al [5], Nakagawa [6] and Pham and Wang [7] emphasized the importance of overhaul.

The model describing the effect of overhaul is fundamental for establishing an appropriate PM policy. Liu et al. [5] and Nakagawa [6] proposed 'virtual age model' and 'reduction model', respectively. The virtual age model assumes that each overhaul decreases hazard rate of a system by a fixed factor, and the reduction model assumes that the hazard rate after overhaul increases more quickly than that before overhaul. The virtual age model is relatively easy to analyze. A weakness of this model is the assumption that an overhaul decreases hazard rate of a system but never changes hazard rate function. The reduction model overcomes this; each overhaul resets the hazard rate to zero (i.e. each overhaul makes the system as good as new). In reality, however, each overhaul may not be able to eliminate all the impending failures. As a result, unlike the replacement, the overhaul cannot make the system as good as new. In other words, the overhaul can only rejuvenate the system and bring the condition of the system to a level somewhere between as good as new and just prior to the overhaul. Since the impending failures not eliminated affect future reliability of the system, the hazard rate function may become higher after each overhaul is performed on the system. Therefore, we propose a model which not only decreases hazard rate of a system to a certain value but also changes hazard rate function after overhaul. We consider the following PM policy: an overhaul is made at periodic times and the system is replaced by a new system at the N th overhaul.

This paper is organized as follows: Section 2 proposes a model describing the effect of the overhaul. The expected cost rates under the proposed model are obtained and numerical studies are performed in the cases of negligible and non-negligible maintenance times in Section 3 and 4, respectively.

The following notations will be used in this paper.

Notations

T	scheduled interval between overhauls
N	scheduled number of overhauls until the system is replaced at time NT
$v_n(T)$	virtual age of the system at n th overhaul
$h_n(t)$	hazard rate in the period of n th overhaul; i.e., during $((n-1)T, nT]$
c_1	cost of minimal repair at failure
c_2	cost of scheduled overhaul
c_3	cost of replacement
$C(N,T)$	expected cost rate of the system
$X_{(n,k)}$	system lifetime after $(k-1)$ th minimal repairs in the period of n th overhaul
$Y_{(n,k)}$	time at k th system failure
$F_{(n,k)}$	distribution function of $Y_{(k,n)}$

R maintenance time

2. The Model

The proposed model is constructed by using the virtual age function and increasing the slope of hazard rate function after overhaul. Figure 1 depicts hazard rate functions before and after the first overhaul. The figure shows that the overhaul decreases the hazard rate but not to zero, and the slope of hazard rate function becomes larger; i.e., the hazard rate right after overhaul is $h_1(\theta T)$ and the slope is the same as $h_2(t)$. Therefore, the hazard rate right after overhaul can be described as $h_2(v_1(T))$, where $v_1(T)$, called virtual age, satisfies $h_2(v_1(T)) = h_1(\theta T)$.

We now derive the virtual age of the system after n th overhaul. Let t_n be the time of n th overhaul, where $t_0 = 0$; that is, $t_n = nT(n = 1, \cdots, N)$. Let $v_n(T)$ be the virtual age right after n th overhaul. The virtual age function of the system is a function of two variables, $V(v,T)$, that specifies the functional relationship between v and T. If the system has hazard rate function $h_n(t)$ in the period of n th overhaul, then the hazard rate of the system is $h_n\{V(v_{n-1}(T),T)\}$ right after t_n. Since the hazard rate function changes to $h_{n+1}(t)$ after n th overhaul, $v_n(T)$ is obtained by formula

$$h_{n+1}(v_n(T)) = h_n\{V(v_{n-1}(T),T)\}. \qquad (1)$$

Kijima et al [3] and Kijima [4] measured the effect of the overhaul on the virtual age by a multiplier $\theta(0 \le \theta \le 1)$, and used virtual age function $V(v,X) = v + \theta X$. Using this virtual age function, virtual age at n th overhaul becomes

$$v_n(T) = h_{n+1}^{-1}\left[h_n\left(v_{n-1}(T) + \theta T\right)\right], \qquad (2)$$

where $v_0(T) = 0$.

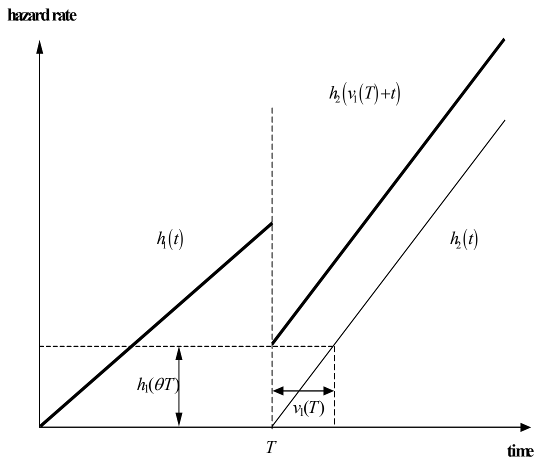

Figure 1 Hazard rate of proposed model

3. Preventive Maintenance Policy - Negligible Maintenance Time

Suppose that the system undergoes minimal repair at failure and maintenance time is zero. Then, failures over time occur according to a nonstationary Poisson process with failure rate $h_n(t)$ in the period of n th overhaul and the expected repair number in $(v_{n-1}(T), v_{n-1}(T)+T]$ is $\int_{v_{n-1}(T)}^{v_{n-1}(T)+T} h_n(t)\,dt$; see, for example, Blischke and Murthy [1]. Therefore, the expected cost in a renewal cycle is $c_1 \sum_{n=1}^{N} \int_{v_{n-1}(T)}^{v_{n-1}(T)+T} h_n(t)\,dt + (N-1)c_2 + c_3$, where c_1 is the cost of minimal repair, c_2 is the cost of overhaul and c_3 is the cost of replacement with $c_3 \ge c_2$, and the expected length of a renewal cycle is NT. Thus the expected cost rate over infinite time horizon is given by

$$C(N,T) = \frac{c_1 \sum_{n=1}^{N} \int_{v_{n-1}(T)}^{v_{n-1}(T)+T} h_n(t)\,dt + (N-1)c_2 + c_3}{NT} \quad (3)$$

The optimal (N,T) minimizing the expected cost rate can be obtained by using a procedure similar to Liu et al. [5] and Nakagawa [6].

Our purpose is to seek both the optimal number of overhauls N^* and the optimal time T^* which minimize $C(N,T)$ in (3). The N^* can be obtained by the inequalities $C(N+1,T) \ge C(N,T)$ and $C(N,T) < C(N-1,T)$, which imply

$$L(N,T) \ge \frac{c_3 - c_2}{c_1} \quad \text{and} \quad L(N-1,T) < \frac{c_3 - c_2}{c_1}, \quad (4)$$

where

$L(N,T)=\begin{cases} N\int_{v_N(T)}^{v_N(T)+T} h_{N+1}(t)dt-\sum_{n=1}^{N}\int_{v_{n-1}(T)}^{v_{n-1}(T)+T} h_n(t)dt & \text{if } N=1,2,\ldots \\ 0 & \text{if } N=0. \end{cases}$ From the assumption that $h_{n+1}(t)>h_n(t)$ for any $t>0$, we obtain

$$L(N,T)-L(N-1,T)=N\left[\int_{v_N(T)}^{v_N(T)+T} h_{N+1}(t)dt-\int_{v_{N-1}(T)}^{v_{N-1}(T)+T} h_N(t)dt\right],$$
$$L(N,T)\geq\int_{v_N(T)}^{v_N(T)+T} h_{N+1}(t)dt-\int_0^T h_1(t)dt. \tag{5}$$

Thus, $L(N,T)$ is increasing in N and, if $\lim_{N\to\infty} h_N(t)\to\infty$ then it tends to ∞ as $N\to\infty$. Therefore, there exists a finite and unique N^* which satisfies (4) for any $T>0$.

Next, differentiating $C(N,T)$ with respect to T and setting it equal to 0, we obtain

$$\sum_{k=1}^{N}\left[\left\{h_n\left(v_{n-1}(T)+T\right)\cdot\left(v_{n-1}'(T)+1\right)-h_n\left(v_{n-1}(T)\right)\cdot v_{n-1}'(T)\right\}\cdot T\right.$$
$$\left.-\int_{v_{n-1}(T)}^{v_{n-1}(T)+T} h_n(t)dt\right]=\frac{\left[(N-1)c_2+c_3\right]}{c_1} \quad . \tag{6}$$

If $h_k(t)$ is differentiable and strictly increasing to ∞, then the left-hand side of (6) strictly increases to ∞. Thus, there exists a finite and unique T^* which satisfies (4) for any integer N.

(N^*,T^*) can be obtained using the following procedure:

i) Let $N_1=1$ and compute $T=T_1$ satisfying $\frac{\partial C(N_1,T)}{\partial T}=0$.

ii) Find $N=N_2$ satisfying the inequalities $C(N+1,T_1)\geq C(N,T_1)$ and $C(N,T_1)<C(N-1,T_1)$.

iii) Compute $T=T_2$ satisfying $\frac{\partial C(N_2,T)}{\partial T}=0$.

iv) If $N_j=N_{j+1}$ $(j=1,2,\ldots)$, set $(N^*,T^*)=(N_j,T_j)$ and stop; otherwise, go to ii).

For the Weibull hazard rate, we obtain $C(N,T)$ and compute the optimal T^*. If the hazard rate in the nth overhaul period is $h_n(t)=\alpha_n\beta t^{\beta-1}$ and virtual age function satisfies formula (2), then

$$v_n(T)=\sum_{k=1}^{n}\left(\frac{\alpha_k}{\alpha_{n+1}}\right)^{\frac{1}{\beta-1}}\cdot\theta T. \tag{7}$$

This can be shown by induction on n. For $n=1$, (7) holds since $\alpha_2\beta\left(v_1(T)\right)^{\beta-1}=\alpha_1\beta(\theta T)^{\beta-1}$, and $v_1(T)=\left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{1}{\beta-1}}\cdot\theta T$. Suppose that

(7) holds for $(n-1)$, $n>1$. With $h_n(t)=\alpha_n\beta t^{\beta-1}$, formula (2) becomes

$$\alpha_{n+1}\beta\left(v_n(T)\right)^{\beta-1}=\alpha_n\beta\left(v_{n-1}(T)+\theta T\right)^{\beta-1}$$

and

$$\begin{aligned} v_n(T) &= \left(\frac{\alpha_n}{\alpha_{n+1}}\right)^{\frac{1}{\beta-1}}\left\{v_{n-1}(T)+\theta T\right\} \\ &= \left(\frac{\alpha_n}{\alpha_{n+1}}\right)^{\frac{1}{\beta-1}}\left[\sum_{k=1}^{n-1}\left(\frac{\alpha_k}{\alpha_n}\right)^{\frac{1}{\beta-1}}+1\right]\cdot\theta T \\ &= \left[\sum_{k=1}^{n-1}\left(\frac{\alpha_k}{\alpha_{n+1}}\right)^{\frac{1}{\beta-1}}+\left(\frac{\alpha_n}{\alpha_{n+1}}\right)^{\frac{1}{\beta-1}}\right]\cdot\theta T \\ &= \sum_{k=1}^{n}\left(\frac{\alpha_k}{\alpha_{n+1}}\right)^{\frac{1}{\beta-1}}\cdot\theta T. \end{aligned}$$

Therefore, (7) holds for every $n\geq 1$.

From (3) and (7),

$$C(N,T)=\frac{c_1R(N,T)+(N-1)c_2+c_3}{NT}, \qquad (8)$$

where $R(N,T)=T^{\beta}\sum_{n=1}^{N}\alpha_n\left[\left\{\sum_{k=1}^{n-1}\left(\frac{\alpha_k}{\alpha_n}\right)^{\frac{1}{\beta-1}}\theta+1\right\}^{\beta}-\left\{\sum_{k=1}^{n-1}\left(\frac{\alpha_k}{\alpha_n}\right)^{\frac{1}{\beta-1}}\theta\right\}^{\beta}\right]$

Numerical Study

Suppose that the time to failure of the system follows Weibull distribution with $\beta=2,3,4$ and $1/\alpha_n=100\times\left(0.9^{\beta}\right)^{n-1}$ $(n=1,2,\text{L})$. That is, the mean time to failure in the nth period of overhaul becomes 10 percent shorter for every overhaul.

Table 1 Optimal maintenance policy for negligible maintenance time

		$\beta=2$			$\beta=3$			$\beta=4$		
θ	c_3/c_1	N^*	T^*	$C(N^*,T^*)$	N^*	T^*	$C(N^*,T^*)$	N^*	T^*	$C(N^*,T^*)$
0.1	3	1	17.32	0.3464	1	5.31	0.8469	1	3.16	1.2649
	10	3	19.06	0.5970	3	5.21	1.5367	3	2.91	2.4460
	20	3	24.29	0.7135	4	5.20	2.0929	4	2.80	3.4525
	50	4	28.95	1.0190	5	5.61	3.3168	5	2.86	5.7833
	100	5	32.68	1.3711	6	5.86	4.9057	6	2.85	8.9668
0.2	3	1	17.32	0.3464	1	5.31	0.8469	1	3.16	1.2649
	10	2	22.21	0.5852	2	5.88	1.6577	2	3.24	2.6723
	20	3	22.68	0.7641	3	5.59	2.3242	3	2.97	3.8930
	50	4	26.44	1.1156	4	5.82	3.7988	4	2.91	6.7509
	100	5	29.33	1.5273	5	5.86	5.7089	5	2.80	10.6584
0.3	3	1	17.32	0.3464	1	5.31	0.8469	1	3.16	1.2649
	10	2	21.42	0.6070	2	5.60	1.7402	2	3.07	2.8222
	20	2	28.49	0.8074	3	5.15	2.5255	3	2.70	4.2776
	50	3	31.35	1.1910	3	6.65	4.2119	3	3.27	7.6052
	100	4	33.29	1.6372	4	6.42	6.3697	4	3.00	12.0924

Table 1 gives the values of N^*, T^*, and the corresponding expected cost rate $C\left(N^*,T^*\right)$ when $\theta=0.1,0.2,0.3$, $c_2/c_1=3$ and $c_3/c_1=3,10,20,50,100$, and shows that:

i) The replacement time and the number of overhauls become larger as replacement cost gets larger.

ii) For a fixed θ, as β gets larger (and hence the system's hazard rate becomes larger), T^* becomes smaller.

iii) For a fixed β, as θ gets larger (and hence overhaul is less effective), the replacement time and the number of overhauls become smaller.

To investigate further the effect of overhaul, additional numerical analysis is performed. Let Δ be the percentage of cost savings of the optimal maintenance policy with overhauls over cost of optimal maintenance policy with only minimal repairs and replacement (which means that there will be no overhauls for the system); that is, $\Delta = 100\times\dfrac{C\left(1,T_1^*\right)-C\left(N^*,T^*\right)}{C\left(1,T_1^*\right)}$. Table 2 and Figure 2 give the optimal N^*,T^*, and the effect of overhaul for $\theta=0.2$ and $c_3/c_1=10$, and show that as the system's hazard rate gets larger, overhaul becomes more effective.

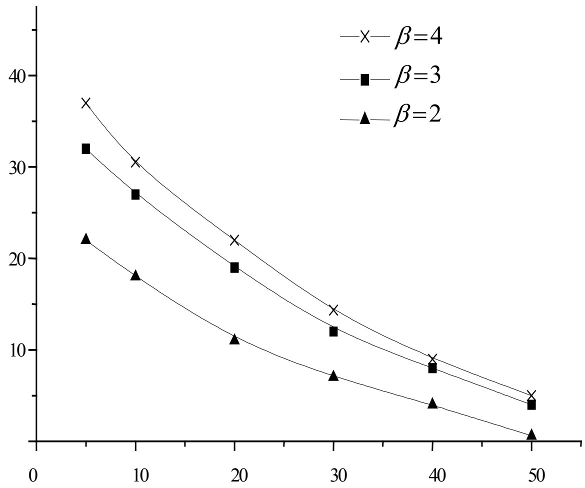

Figure 2 Percentage of cost saving by overhaul

Table 2. Optimal maintenance policy and cost saving for selected values of overhaul cost

	$\beta=2$				$\beta=3$				$\beta=4$			
$(c_2/c_3)\times 100(\%)$	N^*	T^*	$C(N^*,T^*)$	Δ (%)	N^*	T^*	$C(N^*,T^*)$	Δ (%)	N^*	T^*	$C(N^*,T^*)$	Δ (%)
5	4	11.67	0.4925	22	4	3.38	1.2771	32	4	1.94	1.9804	37
10	3	15.41	0.5191	18	4	3.52	1.3858	27	4	2.00	2.1711	31
20	3	16.65	0.5607	11	3	4.55	1.5383	19	3	2.54	2.4471	22
30	2	22.21	0.5852	7	2	5.88	1.6577	13	2	3.24	2.6723	14
40	2	23.05	0.6073	4	2	6.03	1.7416	8	2	3.30	2.8250	9
50	2	23.86	0.6286	0.6	2	6.17	1.8236	4	2	3.36	3.1205	5

Estimation of the repair cost for a complex system is laborious, so that the effects of incorrect estimate of repair cost should be investigated. The percentage error of $C\left(N^*,T^*\right)$ caused by incorrect estimate of repair cost is defined as

$$PE=\frac{C\left(N',T'\right)-C\left(N^*,T^*\right)}{C\left(N^*,T^*\right)}\times 100(\%) \qquad (9)$$

where $C(N^*,T^*)$ is the minimal cost obtained with the correct repair cost and $C(N',T')$ is the cost obtained with incorrect estimate of repair cost.

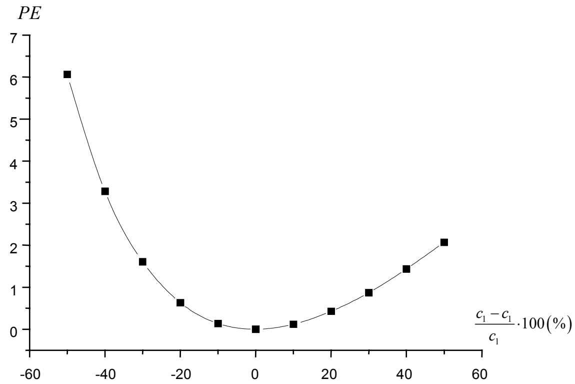

Figure 3 Percentage errors by wrong repair cost

Figure 3 gives PE versus $\frac{c_1 - c_1}{c_1} \cdot 100(\%)$ for $\beta = 2, \theta = 0.2, c_2/c_1 = 3$, and $c_3/c_1 = 10$, and shows that underestimating repair cost causes a larger PE increase than overestimating repair cost.

4. Preventive Maintenance Policy - Non-negligible Maintenance Time

It is assumed that the system undergoes only minimal repair at failure and each minimal repair takes constant time R. The maximum possible number of minimal repair in $(0,t)$ is $\lceil (t/R)+1 \rceil$. In the period of n th overhaul, it follows that

$$\begin{aligned} F_{(n,k)}(t) &= \Pr\left\{ \sum_{j=1}^{k} X_{(n,j)} \le T - (k-1)R \right\} \\ &= \Pr\left\{ N\left(T-(k-1)R\right) \ge k \right\} \end{aligned} . \qquad (10)$$

$\{N(x); x \ge 0\}$ forms a non-homogeneous Poisson process with intensity $h_n(t)$, and

$$F_{(n,k)}(t) = \begin{cases} 0 & , \quad t < (k-1)R \\ 1 - \exp\left[-H_n\left(t-(k-1)R\right)\right] & \\ \quad \times \sum_{j=0}^{k-1} \dfrac{\left[-H_n\left(t-(k-1)R\right)\right]^j}{j!}, & t \ge (k-1)R \end{cases}$$

(11)

See, for example, Dagpunar[2].

The probability of k failures occurring in $(0,t)$ is

$$p_n(t,k) = F_{(n,k)}(t) - F_{(n,k+1)}(t), \qquad (12)$$

where $F_{(n,0)}(t) = 1$.

Then the hazard rate $h_n(t,R)$ considering minimal repair time R is expressed in terms of $h_n(t)$ and $p_n(t,k)$ as follows.

Conditional on the system having survived until t and subjected to k failures in $(0,t]$, the probability that it will fail in $(t,t+\delta t]$ is given by the conditional hazard rate $h_n(t,R|k)$. On removing the conditioning over k, we have

$$\begin{aligned} h_n(t,R) &= \sum_{k=0}^{\lceil (T/R)+1 \rceil} h_n(t,R|k) \cdot p_n(t,k) \\ &= h_n(t) \cdot p_n(t,0) + \sum_{k=1}^{\lceil (T/R)+1 \rceil} \Big[h_n(t-kR) \cdot G_{(n,k)}(t) \\ &\quad + \int_0^R h_n(t-kR+x) \cdot g_{(n,k)}(t+x)\,dx \Big] \cdot p_n(t,k) \quad , \end{aligned}$$

(13)

where $G_{(n,k)}(x) = \dfrac{p_n(x-R,k)}{p_n(t,k)}$.

The expected cost rate of formula (3) now becomes

$$C(N,T) = \frac{c_1 \sum_{n=1}^{N} \int_{v_{n-1}(T)}^{v_{n-1}(T)+T} h_n(t,R)\,dt + (N-1)c_2 + c_3}{N(T+R)} . \qquad (14)$$

Since the optimal solution (N^*,T^*) cannot be obtained analytically, numerical methods such as Powell algorithm and golden section search method are used.

Numerical Study

Suppose that the time to failure of the system follows Weibull distribution with $\beta = 2$, $1/\alpha_n = 100 \times (0.81)^{n-1}$ $(n = 1,2,\ldots)$, and $\theta = 0.2$, $c_1/c_2 = 3$, $c_3/c_2 = 3,10,20,50$. Table 3 gives the values of N^*, T^*, and the corresponding expected cost rate $C(N^*,T^*)$ when $R = 0.5, 1, 2$.

Table 3 shows that as maintenance time increases replacement time increases.

5. Conclusions

We have proposed an improved model for describing a system subject to minimal repair and overhaul and established optimal maintenance policies in the cases of negligible and non-negligible maintenance time. Cost models are constructed considering minimal repair, overhaul and replacement. Numerical studies show that overhaul becomes more effective as the system's hazard rate gets larger, and underestimating repair cost is more serious than overestimating it.

A possible area of further investigation would be to consider availability in the case of non-negligible maintenance time, and to relax the assumption of constant maintenance time.

Table 3. Optimal maintenance policy for non-negligible maintenance time

	$R=0.5$			$R=1.0$			$R=2.0$		
c_3/c_1	N^*	T^*	$C(N^*,T^*)$	N^*	T^*	$C(N^*,T^*)$	N^*	T^*	$C(N^*,T^*)$
3	1	17.42	0.3454	1	18.86	0.3318	1	19.66	0.3250
10	2	24.02	0.5625	2	25.38	0.5470	2	27.20	0.5298
20	3	24.58	0.7297	3	26.20	0.7112	3	28.65	0.6845
50	4	27.77	1.0571	4	28.64	1.0381	4	30.69	1.0211

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